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Mean field theory for $su(4) \simeq so(6)$

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Abstract
Algebraic mean field theory constructs a group theoretical model of quantum systems that have a weak dynamical symmetry but may break dynamical symmetry. The strong defining condition for dynamical symmetry is that states belong to one irreducible representation space. Weak dynamical symmetry demands that the densities corresponding to the states have a constant value for each Casimir. Quantum phase transitions and other complex systems exhibit weak dynamical symmetry. Furthermore mean field theory often yields analytic formulae for expectations and energy spectra that are not feasible in representation theory. This paper develops mean field theory on any coadjoint orbit of $su(4)$ densities. The simple Lie algebra $su(4) \simeq so(6)$ is a 15-dimensional algebra that contains the subalgebra $usp(4) \simeq so(5)$ and the angular momentum algebra $su(2)$. The $su(4)$ dual space consists of density matrices which are defined by the expectations of the $su(4)$ generators. A coadjoint orbit is a common level surface in the dual space of the three $su(4)$ Casimirs. A Lax pair determines the dynamics of these densities on each coadjoint orbit. Analytic solutions are reported for rotating $su(4)$ densities in equilibrium for a particular energy function.

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1. Introduction

This paper applies the algebraic mean field method to the 15-dimensional simple Lie algebra $su(4) \cong so(6)$. The mean field method constructs a group theoretical model of quantum many-body systems that satisfy the *weak* dynamical symmetry assumption which is defined below. The system is said to have a dynamical symmetry if the quantum states are vectors from one irreducible representation (irrep) space. Although any dynamical symmetry is also a weak dynamical symmetry, the converse is not true. In prior work the algebraic mean field method was applied in the field of nuclear structure physics to the Elliott $su(3)$ model [1–5], the symplectic $sp(3, \mathrm{R})$ collective model [6–9], and the gcm(3) general collective motion or

Riemann ellipsoid model [10]. The mean field theory of the $usp(4) \cong so(5)$ subalgebra [11] of $su(4) \cong so(6)$ is closely related to the present work.

Let G denote a Lie group with Lie algebra $\mathfrak{g}$. Each element of $\mathfrak{g}$ is assumed to be associated uniquely with a physical observable. Let $\mathfrak{g}^*$ be the vector space dual to the Lie algebra $\mathfrak{g}$ and denote by $\langle \rho, S \rangle \in \mathbf{R}$ the value of the linear functional $\rho \in \mathfrak{g}^*$ at $S \in \mathfrak{g}$. Each point ρ of the dual space is interpreted physically as a $\mathfrak{g}$-density, or just a density, which determines the quantum expectation $\langle \rho, S \rangle$ of the observable corresponding to the Lie algebra element S. When $\mathfrak{g}$ is a semisimple matrix algebra, like $su(4) \cong so(6)$, $\mathfrak{g}^*$ may be identified with $\mathfrak{g}$ and the pairing $\langle \rho, S \rangle \propto \mathrm{tr}(\rho S)$ is proportional to the nondegenerate Killing form. Moreover, in this case, the adjoint action of the group G on $\mathfrak{g}$, $\mathrm{Ad}_g S = gSg^{-1}$, induces the coadjoint action of G on $\mathfrak{g}^*$, $\mathrm{Ad}^*_g \rho = g\rho g^{-1}$, for all $g \in G$, $\rho \in \mathfrak{g}^*$, $S \in \mathfrak{g}$.

A unitary representation π, not necessarily irreducible, of $\mathfrak{g}$ on the Hilbert space $\mathcal{H}$ of quantum many-body states provides the algebra's physical interpretation. Corresponding to each quantum state $\Psi \in \mathcal{H}$, there is a unique $\mathfrak{g}$-density ρ such that $\langle \rho, S \rangle = \langle \Psi | \pi(S) \Psi \rangle / \langle \Psi | \Psi \rangle$ for all $S \in \mathfrak{g}$. The coadjoint action is compatible with the representation in the sense that the density corresponding to the state $\pi(g)\Psi$ is $\mathrm{Ad}^*_g \rho$ for all $g \in G$.

The coadjoint orbit $\mathcal{O}_\rho$ containing the density ρ is the manifold

$$\mathcal{O}_\rho = \{\mathrm{Ad}^*_g \rho \mid g \in G\}. \tag{1}$$

The $\mathfrak{g}$-densities corresponding to the coherent states, $\{\pi(g)\Psi \mid g \in G\}$, generated from a quantum state vector Ψ in a representation space $\mathcal{H}$ lie on one coadjoint orbit of G. When the densities corresponding to a subset of quantum states lie on one coadjoint orbit, the system is said to have a *weak* dynamical symmetry.

The algebra's Casimirs are real-valued coadjoint-invariant functions on the dual space. Each common level surface of the Casimir functions is a union of a finite number of orbits, and a generic level is just one coadjoint orbit [12, 13]. The possible existence of a weak dynamical symmetry for some set of quantum states can be put to the test by evaluating the Casimir functions. An irrep space is an eigenspace for each Casimir operator, and, therefore, every dynamical symmetry determines a weak dynamical symmetry.

An interesting example of a weak dynamical symmetry that is not a dynamical symmetry is provided by a quantum phase transition in the interacting boson model [14–20]. The quantum state space $\mathcal{H}$ is a symmetric irrep of $u(6)$ determined by the boson number N. Let $\hat{H}_{u5}$, respectively $\hat{H}_{su3}$, denote Casimir operators for the $u(5)$, respectively $su(3)$, subalgebras of $u(6)$, and define the Hamiltonian $\hat{H}_\alpha = \alpha \hat{H}_{su3} + (1-\alpha)\hat{H}_{u5}$. For large N a quantum phase transition exists at a critical value $\alpha_{\mathrm{cr}} \approx 0.5$. For $\alpha < \alpha_{\mathrm{cr}}$ the system is in a $u(5)$ phase, while for $\alpha > \alpha_{\mathrm{cr}}$ the system is in an $su(3)$ phase. More precisely, for $\alpha > \alpha_{\mathrm{cr}}$, the densities of the low-energy eigenstates of $\hat{H}_\alpha$ in the large N limit define a weak $su(3)$ dynamical symmetry although these eigenstates are not even approximately vectors from one $su(3)$ irrep space [21].

An algebraic mean field theory is defined by a triple $(G, \mathcal{O}, \mathcal{E})$ where G is a Lie group, $\mathcal{O}$ is a coadjoint orbit in the dual space of the Lie algebra $\mathfrak{g}$ and $\mathcal{E}$ is a smooth real-valued energy function on the dual space. Each coadjoint orbit is a symplectic manifold [22], and the energy function defines a Hamiltonian vector field on this manifold [23]. This vector field may be viewed as a $\mathfrak{g}$-valued function $h[\rho]$ of the density ρ. Mean field dynamics is the Hamiltonian dynamical system defined by this vector field. For a matrix Lie group, the mean field dynamical system simplifies to a finite-dimensional Lax system [24, 25], $\dot{\rho} = \mathrm{i}[\rho, h[\rho]]$. Mean field dynamics is compatible with the Schrödinger equation on the Hilbert space of states. The zeros of the Hamiltonian vector field are the equilibrium densities.

Geometric quantization is a scheme for constructing irreps starting from one coadjoint orbit. Kirillov invented the coadjoint orbit method for nilpotent Lie groups [26], but he, Kostant, Souriau, Vogan and many others have extended the construction to solvable and semisimple groups [28–31]. To each coadjoint orbit a natural representation, known as prequantization, is associated whenever the orbit satisfies a certain integrality condition. For a simple compact Lie group, the integrality condition requires that the coadjoint orbit contains the density corresponding to a highest weight vector. Various ideas from the theory of geometric quantization or coadjoint orbits are relevant to mean field theory, for example, the Kirillov program to determine properties of irreps directly from the coadjoint orbit data [32]. However, mean field theory is not restricted to integral orbits.

Section 2 defines the matrix Lie algebra $su(4) \simeq so(6)$ and assumes a unitary representation π of it is given. The representation determines the physical interpretation. It might be associated in nuclear structure physics with the $so(6)$ dynamical symmetry limit of the interacting boson model [33] or the fermion dynamical symmetry model [34]. Possible applications in solid state physics include the one-dimensional spin–orbital model [35, 36], the spin-ladder model [37] and the coexistence of superconductivity and charge-density waves [38]. The general theory for $su(4)$ mean fields developed in this paper may be applied to any of these topics in many-body physics.

To set up the mean field theory of the matrix algebra $su(4)$, this paper carries out a sequence of well-defined steps. Section 3 determines explicitly the dual space $su(4)^*$, consisting of all $su(4)$ density matrices, and the coadjoint action of the Lie group $SU(4)$ on the dual space. The set of all coadjoint orbits is enumerated next. The $su(4)$ Casimirs are constant real-valued functions on each coadjoint orbit. Among the level surfaces of the Casimir functions are the integral coadjoint orbits associated with the highest weight irreducible representations of $su(4) \simeq so(6)$.

Section 4 defines the symplectic geometry of a coadjoint orbit. This geometry associates a Hamiltonian vector field with each smooth function on a coadjoint orbit. This section reports the Hamiltonian vector fields associated with several functions of physical interest, including the energy function. When the energy function is rotationally invariant, the dynamics on a coadjoint orbit simplifies to a Lax system on a proper submanifold of the orbit space. Section 4 also reports analytic equilibrium solutions for a particular rotational scalar energy function.

2. Algebra definition

Suppose $\hat{L}^{(1)}_\mu, \mu = 0, \pm 1$, denote the spherical components of an angular momentum tensor operator, spanning the Lie algebra $su(2)$, $\hat{Q}^{(2)}_\mu, \mu = 0, \pm 1, \pm 2$, are the components of a quadrupole tensor operator and $\hat{O}^{(3)}_\mu, \mu = 0, \pm 1, \pm 2, \pm 3$, denote the components of an octupole tensor operator. These (dimensionless) operators are assumed to be Hermitian when acting on a Hilbert space $\mathcal{H}$ and to obey the commutation relations,

$$[\hat{Q}^{(2)} \times \hat{Q}^{(2)}]^{(k)} = \begin{cases} \frac{\sqrt{10}}{2}\hat{L}^{(1)}, & k = 1 \\ -\frac{\sqrt{10}}{2}\hat{O}^{(3)}, & k = 3 \end{cases} \tag{2}$$

$$[\hat{O}^{(3)} \times \hat{O}^{(3)}]^{(k)} = \begin{cases} -\sqrt{7}\hat{L}^{(1)}, & k = 1 \\ -\frac{\sqrt{6}}{2}\hat{O}^{(3)}, & k = 3 \\ 0, & k = 5, \end{cases} \tag{3}$$

and the additional commutators,

$$[\hat{O}^{(3)} \times \hat{Q}^{(2)}]^{(k)} + (-1)^k[\hat{Q}^{(2)} \times \hat{O}^{(3)}]^{(k)} = \begin{cases} 0 & k = 1, 3, 4, 5 \\ \sqrt{14}\hat{Q}^{(2)} & k = 2. \end{cases} \tag{4}$$

The brackets on the left sides denote angular momentum coupled tensors.

The algebra of operators $\{\hat{L}^{(1)}_\mu, \hat{Q}^{(2)}_\mu, \hat{O}^{(3)}_\mu\}$ does not necessarily act irreducibly on $\mathcal{H}$. In either the Bohr–Mottelson collective model or the interacting boson model, such operators are given by $\hat{L}^{(1)} = \sqrt{10}[d^\dagger \times \tilde{d}]^{(1)}$, $\hat{Q}^{(2)} = d^\dagger s + s^\dagger \tilde{d}$ and $\hat{O}^{(3)} = -\sqrt{10}[d^\dagger \times \tilde{d}]^{(3)}$, where $d^\dagger$ and $\tilde{d}$ denote the creation and destruction tensor operators for the spherical components of a d-boson, and $s^\dagger$, s, the scalar s-boson operators.

The set of 15 operators $\{\hat{L}^{(1)}_\mu, \hat{Q}^{(2)}_\mu, \hat{O}^{(3)}_\mu\}$ closes under commutation to form a Lie algebra that will be shown now to be a unitary representation of $su(4) \cong so(6)$. The matrix Lie algebra $su(4)$ is defined by

$$su(4) = \{S \in M_4(C) \mid S^\dagger = S, \operatorname{tr} S = 0\}, \tag{5}$$

$M_4(C)$ denotes the algebra of 4×4 complex, matrices and $S^\dagger$ denotes the Hermitian conjugate of the matrix S. Strictly speaking, the elements of the real Lie algebra $su(4)$ should be skew-Hermitian matrices instead of Hermitian matrices, but the correspondence with physics is enhanced using Hermitian S. The unitary group is the connected and simply connected matrix Lie group,

$$SU(4) = \{g \in M_4(C) \mid g^\dagger \cdot g = \mathrm{Id}, \det g = 1\}, \tag{6}$$

where Id denotes the identity matrix. When S is a Lie algebra element in $su(4)$, its exponentiation $\exp(\mathrm{i}S)$ is a matrix in the Lie group $SU(4)$. The subalgebra of operators spanned the vector and octupole operators is isomorphic to the ten-dimensional algebra $usp(4) \cong so(5)$.

A basis for the complexification of the real Lie algebra $su(4)$ is the set of 15 matrices $\{\mathcal{L}_\mu, \mathcal{Q}_\mu, \mathcal{O}_\mu\}$, which are defined in the second column of table 1. E_{ij} denotes the elementary matrix all of whose entries are zero except for the entry equal to 1 at the intersection of row i and column j. The $su(4)$ basis matrices satisfy the identities, $(\mathcal{L}_\mu)^\dagger = (-1)^\mu \mathcal{L}_{-\mu}$, $(\mathcal{Q}_\mu)^\dagger = (-1)^\mu \mathcal{Q}_{-\mu}$ and $(\mathcal{O}_\mu)^\dagger = (-1)^\mu \mathcal{O}_{-\mu}$. A general element $S(u, v, w)$ of $su4)$ is defined by a set of 15 complex numbers satisfying $u_{-\mu} = (-1)^\mu (u_\mu)^*$, $\mu = 0, \pm 1$, $v_{-\mu} = (-1)^\mu (v_\mu)^*$, $\mu = 0, \pm 1, \pm 2, \pm 3$ and $w_{-\mu} = (-1)^\mu (w_\mu)^*$, $\mu = 0, \pm 1, \pm 2$,

$$S(u, v, w) = u \cdot \mathcal{L} + v \cdot \mathcal{O} + w \cdot \mathcal{Q}, \tag{7}$$

where $u \cdot \mathcal{L} = \sum_{\mu=-1}^{1} (-1)^\mu u_{-\mu} \mathcal{L}_\mu$ and similarly for $v \cdot \mathcal{O}$ and $w \cdot \mathcal{Q}$. The compact real Lie algebras $su(4)$ and $so(6)$ are isomorphic because of the low-dimensional Cartan isomorphism $A_3 \cong D_3$.

For $S = S(u, v, w)$ in $su(4)$, define the operator

$$\pi(S) = \sum_{\mu=-1}^{1} u_\mu \hat{L}_\mu + \sum_{\mu=-2}^{2} w_\mu \hat{Q}_\mu + \sum_{\mu=-3}^{3} v_\mu \hat{O}_\mu. \tag{8}$$

π determines a unitary representation of $su(4)$ since $\pi(S)^\dagger = \pi(S)$ and $\pi([S_1, S_2]) = [\pi(S_1), \pi(S_2)]$ for $S_1, S_2 \in su(4)$. π extends to a representation of the complexification of $su(4)$ with $\pi(\mathcal{L}_\mu) = (-1)^\mu \hat{L}_{-\mu}$, $\pi(\mathcal{Q}_\mu) = (-1)^\mu \hat{Q}_{-\mu}$, and $\pi(\mathcal{O}_\mu) = (-1)^\mu \hat{O}_{-\mu}$. The ten-dimensional $usp(4) \cong so(5)$ subalgebra consists of the matrices $S(u, v, w = 0)$.

The subalgebra $su(2)$ of $su(4)$ consists of the matrices $S(u, v = 0, w = 0)$, which is the four-dimensional $j = 3/2$ representation of the Lie algebra of the rotation group. The $SU(2)$ subgroup of $SU(4)$ consists of the 4×4 unitary matrices $R = \mathcal{D}^{(3/2)}(\alpha, \beta, \gamma)$, where α, β, γ are the Euler angles.

The rotation group $SU(2)$ acts on the Lie algebra $su(4)$ by the adjoint transformation, $\mathrm{Ad}_R S = R \cdot S \cdot R^{-1}$. Since $\hat{L}^{(1)}$, $\hat{Q}^{(2)}$ and $\hat{O}^{(3)}$ are irreducible tensor operators and π is a representation, a rotated algebra element is represented by $\mathrm{Ad}_R S(u, v, w) = S(\mathcal{D}^{(1)}(R)u, \mathcal{D}^{(3)}(R)v, \mathcal{D}^{(2)}(R)w)$.

Table 1. Hamiltonian vector fields corresponding to elementary functions and a basis for the complexification of $su(4)$.

Function f	Hamiltonian vector field $S_f[\rho]$
$\lambda(S)$	S
l_0	$\mathcal{L}_0 = \frac{3}{2}E_{11} + \frac{1}{2}E_{22} - \frac{1}{2}E_{33} - \frac{3}{2}E_{44}$
l_1	$\mathcal{L}_1 = -\sqrt{\frac{3}{2}}E_{21} - \sqrt{2}E_{32} - \sqrt{\frac{3}{2}}E_{43}$
l_{-1}	$\mathcal{L}_{-1} = \sqrt{\frac{3}{2}}E_{12} + \sqrt{2}E_{23} + \sqrt{\frac{3}{2}}E_{34}$
q_0	$\mathcal{Q}_0 = \frac{\sqrt{5}}{2}(E_{11} - E_{22} - E_{33} + E_{44})$
q_1	$\mathcal{Q}_1 = -\sqrt{\frac{5}{2}}E_{21} + \sqrt{\frac{5}{2}}E_{43}$
q_{-1}	$\mathcal{Q}_{-1} = \sqrt{\frac{5}{2}}E_{12} - \sqrt{\frac{5}{2}}E_{34}$
q_2	$\mathcal{Q}_2 = \sqrt{\frac{5}{2}}E_{31} + \sqrt{\frac{5}{2}}E_{42}$
q_{-2}	$\mathcal{Q}_{-2} = \sqrt{\frac{5}{2}}E_{13} + \sqrt{\frac{5}{2}}E_{24}$
o_0	$\mathcal{O}_0 = \frac{1}{2}E_{11} - \frac{3}{2}E_{22} + \frac{3}{2}E_{33} - \frac{1}{2}E_{44}$
o_1	$\mathcal{O}_1 = -E_{21} + \sqrt{3}E_{32} - E_{43}$
o_{-1}	$\mathcal{O}_{-1} = E_{12} - \sqrt{3}E_{23} + E_{34}$
o_2	$\mathcal{O}_2 = \sqrt{\frac{5}{2}}E_{31} - \sqrt{\frac{5}{2}}E_{42}$
o_{-2}	$\mathcal{O}_{-2} = \sqrt{\frac{5}{2}}E_{13} - \sqrt{\frac{5}{2}}E_{24}$
o_3	$\mathcal{O}_3 = -\sqrt{5}E_{41}$
o_{-3}	$\mathcal{O}_{-3} = \sqrt{5}E_{14}$

3. Density matrices

Given a state vector $|\Psi\rangle$ in the representation space $\mathcal{H}$, the expectations of the $su(4)$ operators are

$$\begin{aligned} l_\mu &= (-1)^\mu \langle\Psi|\hat{L}^{(1)}_{-\mu}|\Psi\rangle = \langle\Psi|\pi(\mathcal{L}_\mu)|\Psi\rangle \\ q_\mu &= (-1)^\mu \langle\Psi|\hat{Q}^{(2)}_{-\mu}|\Psi\rangle = \langle\Psi|\pi(\mathcal{Q}_\mu)|\Psi\rangle \\ o_\mu &= (-1)^\mu \langle\Psi|\hat{O}^{(3)}_{-\mu}|\Psi\rangle = \langle\Psi|\pi(\mathcal{O}_\mu)|\Psi\rangle. \end{aligned} \tag{9}$$

Note that $l_{-\mu} = (-1)^\mu (l_\mu)^*$ and similarly for q_μ and o_μ. When $|\Psi\rangle$ is rotated to $\pi(R)|\Psi\rangle$, the angular momentum expectation transforms from l_μ to $l_{\mu'} = \sum_{\mu=-1}^{1} \mathcal{D}^{(1)}_{\mu',\mu}(R) l_\mu$ and similarly for q_μ and o_μ. Therefore these expectations l, q and o are components of vector, quadrupole and octupole spherical tensors.

Define the density matrix

$$\rho(l, o, q) = l \cdot \mathcal{L} + o \cdot \mathcal{O} + q \cdot \mathcal{Q}. \tag{10}$$

The matrix ρ is an element of the algebra's dual space. The pairing between a density $\rho \in su(4)^*$ and an algebra element $S \in su(4)$ is defined by

$$\langle\rho, S\rangle = \tfrac{1}{5}\,\mathrm{tr}(\rho S), \tag{11}$$

and this real number equals the expectation of the operator $\pi(S)$ in the state $|\Psi\rangle$,

$$\begin{aligned} \langle\rho, S\rangle &= l \cdot u + o \cdot v + q \cdot w \\ &= \langle\Psi|\pi(S)|\Psi\rangle. \end{aligned} \tag{12}$$

When π is an irreducible representation of $su(4)$ and $|\Psi\rangle$ is a highest weight vector, the density is a diagonal matrix,

$$\varrho = \tfrac{5}{4}\,\mathrm{diag}(3m_1+2m_2+m_3, -m_1+2m_2+m_3, -m_1-2m_2+m_3, -m_1-2m_2-3m_3), \quad (13)$$

where m_1, m_2, m_3 are the nonnegative integral weights that label π.

$$m_1 = \langle \varrho, E_{11} - E_{22}\rangle \qquad m_2 = \langle \varrho, E_{22} - E_{33}\rangle \qquad m_3 = \langle \varrho, E_{33} - E_{44}\rangle. \quad (14)$$

3.1. Coadjoint orbits

The special unitary group acts on its Lie algebra by the adjoint transformation, $\mathrm{Ad}_g S = g\cdot S\cdot g^{-1}$ for $S \in su(4)$ and $g \in SU(4)$. This induces the coadjoint action of $SU(4)$ on the dual space $su(4)^*$, $\mathrm{Ad}^*_g\rho = g\cdot\rho\cdot g^{-1}$ for $\rho \in su(4)^*$ and $g \in SU(4)$. The coadjoint and adjoint actions are related by the pairing, $\langle \mathrm{Ad}^*_g\rho, S\rangle = \langle \rho, \mathrm{Ad}_{g^{-1}} S\rangle$. The coadjoint orbit containing the point ρ consists of ρ and all the transformed densities $\mathrm{Ad}^*_g\rho$ as g ranges over the group $SU(4)$. The mean field approximation limits the model densities to one coadjoint orbit.

Each coadjoint orbit contains a real diagonal traceless matrix ϱ. When the order of the eigenvalues is fixed, the coadjoint orbits are enumerated uniquely by

$$\mathcal{O}_\varrho = \{\mathrm{Ad}^*_g\varrho \mid g \in SU(4)\}, \quad (15)$$

where ϱ is given by equation (13) with m_1, m_2, m_3 being nonnegative real numbers. The orbits $\mathcal{O}_\varrho$ do not intersect when m_1, m_2, m_3 are restricted to nonnegative real numbers. The integral orbits are those with m_1, m_2, m_3 being nonnegative integers. The geometric quantization method or the Borel–Weil theorem associates naturally an irreducible unitary representation of $SU(4)$ with each integral coadjoint orbit.

Each coadjoint orbit $\mathcal{O}_\varrho$ is diffeomorphic to a homogeneous space that equals the group $SU(4)$ modulo the isotropy subgroup H_ϱ at ϱ. The isotropy subgroup consists of the $SU(4)$ group elements g that commute with ϱ. There are six possibilities depending on m_1, m_2, m_3,

$$H_\varrho = \begin{cases} U(1)\times U(1)\times U(1), & m_1, m_2, m_3 > 0 \\ U(1)\times U(2), & \text{exactly one of the } m_k = 0 \\ SU(2)\times U(2) & m_1 = m_3 = 0, m_2 \neq 0 \\ U(3) & m_1 = m_2 = 0, m_3 \neq 0 \\ U(3) & m_1 \neq 0, m_2 = m_3 = 0 \\ SU(4), & m_1 = m_2 = m_3 = 0. \end{cases} \quad (16)$$

The dimension of a homogeneous space, $SU(4)/H_\varrho$, is the difference between the dimension of $SU(4)$ and the dimension of the isotropy subgroup,

$$\dim \mathcal{O}_\varrho = \begin{cases} 12 & m_1, m_2, m_3 > 0 \\ 10, & \text{exactly one of the } m_k = 0 \\ 8, & m_1 = m_3 = 0, m_2 \neq 0 \\ 6, & m_1 = m_2 = 0, m_3 \neq 0 \\ 6 & m_1 \neq 0, m_2 = m_3 = 0 \\ 0, & m_1 = m_2 = m_3 = 0. \end{cases} \quad (17)$$

The Casimir invariants,

$$\mathcal{C}^{(n)}(\rho) = \tfrac{1}{5}\,\mathrm{tr}(\rho^n), \quad (18)$$

are real-valued constant functions on each coadjoint orbit, $\mathcal{C}^{(n)}(\mathrm{Ad}^*_g\rho) = \mathcal{C}^{(n)}(\rho)$ for $g \in SU(4)$ and $\rho \in su(4)^*$. The value of a Casimir may be computed most easily at the orbit representative ϱ. For $n = 2, 3, 4$ the Casimirs equal

$$\mathcal{C}^{(n)}(\varrho) = \frac{1}{5}\left(\frac{5}{4}\right)^n[(3m_1 + 2m_2 + m_3)^n + (-m_1 + 2m_2 + m_3)^n + (-m_1 - 2m_2 + m_3)^n + (-m_1 - 2m_2 - 3m_3)^n]. \tag{19}$$

The Casimirs of higher degree $n \geqslant 5$ are not functionally independent of the quadratic, cubic and quartic invariants.

After expressing the density matrix explicitly as $\rho = \rho(l, o, q)$, the quadratic Casimir is calculated to be

$$\mathcal{C}^{(2)}(\rho) = l \cdot l + o \cdot o + q \cdot q. \tag{20}$$

Expressions for the Casimirs of order 3 and 4 are rather complicated and unilluminating, but are easy to derive with the aid of a computer algebra program.

For a compact Lie group a coadjoint orbit is identical to a common level surface of its Casimir functions. The characterization of a coadjoint orbit as a level surface is especially useful for computations. This level surface in the dual space $su(4)^*$ is an algebraic surface defined by three polynomial Casimir equations in the complex variables l_μ, o_μ and q_μ that make up a density matrix

$$\mathcal{C}^{(n)}(\rho) = \mathcal{C}^{(n)}(\varrho), \qquad \text{for} \quad n = 2, 3, 4. \tag{21}$$

4. Symplectic geometry

This section defines the symplectic geometry on each coadjoint orbit $\mathcal{O}_\varrho$ and uses this structure to associate a Hamiltonian vector field with each smooth real-valued function on $\mathcal{O}_\varrho$. The symplectic form ω_ρ at any point ρ of the orbit is a closed, nondegenerate 2-form that is equivalent to a Poisson bracket. This form determines the dynamics of $su(4)$ densities from a given energy function.

Suppose ρ is any point on a coadjoint orbit $\mathcal{O}_\varrho$. Every Lie algebra element S defines a vector $\bar{S}$ at each point ρ that is tangent to the surface $\mathcal{O}_\varrho$ in the dual space: $\bar{S}$ denotes the tangent to the curve $\epsilon \mapsto \exp(-\mathrm{i}\epsilon S) \cdot \rho \cdot \exp(\mathrm{i}\epsilon S)$. When S is in the annihilator $\mathcal{A}_\rho$ at ρ, $[S, \rho] = 0$, the vector $\bar{S}$ vanishes. Two tangent vectors $\bar{S}$ and $\bar{T}$ are equal when $S - T \in \mathcal{A}_\rho$. Thus the tangent space to the orbit $\mathcal{O}_\varrho$ at ρ may be identified with the coset space $su(4)/\mathcal{A}_\rho$.

For $S, T \in su(4)$, define the symplectic form at ρ by

$$\omega_\rho(\bar{S}, \bar{T}) = -\mathrm{i}\langle \rho, [S, T]\rangle. \tag{22}$$

This form is well defined since, if $S - S' \in \mathcal{A}_\rho$ and $T - T' \in \mathcal{A}_\rho$, then $\langle \rho, [S, T]\rangle = \langle \rho, [S', T']\rangle$. The form is evidently antisymmetric. Moreover, because $su(4)$ is a simple Lie algebra with a nondegenerate Killing form, ω_ρ is likewise nondegenerate, i.e., $\omega_\rho(\bar{S}, \bar{T}) = 0$ for all tangents $\bar{T}$ at ρ implies that $\bar{S}$ vanishes. As a consequence of the Jacobi identity, this form is also closed. These various facts about ω_ρ and the symplectic geometry on any coadjoint orbit $\mathcal{O}_\varrho$ are well known [32].

Given any smooth real-valued function f on the orbit $\mathcal{O}_\varrho$ there exists a vector field $\bar{S}_f$ on the orbit surface such that

$$\begin{aligned}\omega_\rho(\bar{S}_f, \bar{T}) &= \mathrm{d}f(\bar{T}) \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\exp(-\mathrm{i}\epsilon T) \cdot \rho \cdot \exp(\mathrm{i}\epsilon T))|_{\epsilon=0},\end{aligned} \tag{23}$$

for all tangents $\bar{T}$ to the orbit surface at ρ. There is a unique solution, $\bar{S}_f$, to this equation because the symplectic form is nondegenerate. Naturally the Lie algebra element S_f is not

Table 2. Hamiltonian vector fields $S(u, v, w) \in su(4)$ corresponding to some smooth functions f of the density.

Function f	u	v	w
$l \cdot l$	$2l$	0	0
$o \cdot o$	0	$2o$	0
$q \cdot q$	0	0	$2q$
$(l \cdot l)^2$	$4(l \cdot l)l$	0	0
$(o \cdot o)^2$	0	$4(o \cdot o)o$	0
$(q \cdot q)^2$	0	0	$4(q \cdot q)q$
$l \cdot [q \times l]^{(1)}$	$[q \times l]^{(1)}$	0	$-\sqrt{\frac{3}{5}}[l \times l]^{(2)}$
$o \cdot [q \times o]^{(3)}$	0	$2[q \times o]^{(3)}$	$-\sqrt{\frac{7}{5}}[o \times o]^{(2)}$
$l \cdot [q \times o]^{(1)}$	$[q \times o]^{(1)}$	$\sqrt{\frac{3}{7}}[l \times q]^{(3)}$	$-\sqrt{\frac{3}{5}}[l \times o]^{(2)}$
$[o \times o]^{(2)} \cdot [o \times o]^{(2)}$	0	$-4\sqrt{\frac{5}{7}}[[o \times o]^{(2)} \times o]^{(3)}$	0
$[l \times [o \times o]^{(2)}]^{(1)} \cdot l$	$2[l \times [o \times o]^{(2)}]^{(1)}$	$2\sqrt{\frac{5}{7}}[[l \times l]^{(2)} \times o]^{(3)}$	0
$\mathcal{C}^{(n)}$	0	0	0

unique as any element of the annihilator at ρ may be added to S_f. $\bar{S}_f$ is called the Hamiltonian vector field associated with f even when f is not the energy. When $f = \mathcal{E}$ is the energy function, the Hamiltonian vector field is the mean field Hamiltonian, $\bar{h} = \bar{S}_{\mathcal{E}}$,

$$\omega_\rho(\bar{h}[\rho], \bar{T}) = \mathrm{d}\mathcal{E}(\bar{T}), \tag{24}$$

for all $T \in su(4)$.

For each $S \in su(4)$, consider the elementary real-valued function on the dual space, $\lambda(S)(\rho) = \langle \rho, S\rangle$. The value of the function $\lambda(S)$ at the density ρ equals the expectation of the observable S with respect to any state whose density equals ρ. It is proven easily that the Hamiltonian vector field associated with the function $\lambda(S)$ is $\bar{S}$. In particular, the Hamiltonian vector fields associated with the 'coordinate' functions l_μ, o_μ and q_μ are $\bar{\mathcal{L}}_\mu, \bar{\mathcal{O}}_\mu$ and $\bar{\mathcal{Q}}_\mu$, respectively, cf, equation (9).

The Hamiltonian vector field associated with a function f on $\mathcal{O}_\varrho$ that is itself a function of the 'coordinate' functions may be computed using the properties of the differential. For example, when $f = l \cdot l$, the differential is

$$\mathrm{d}f = \sum_\mu (-1)^\mu (\mathrm{d}l_{-\mu} l_\mu + l_{-\mu}\,\mathrm{d}l_\mu) = 2\sum_\mu (-1)^\mu l_{-\mu}\,\mathrm{d}l_\mu. \tag{25}$$

The Hamiltonian vector field is

$$\bar{S}_{l\cdot l} = 2l \cdot \bar{\mathcal{L}}. \tag{26}$$

The Hamiltonian vector fields associated with various smooth functions are provided in table 2.

The Casimirs are constant functions on each coadjoint orbit and their differentials must, therefore, vanish. To verify this, note that the Hamiltonian vector field associated with the quadratic Casimir, equation (20), is $2(l \cdot \mathcal{L} + o \cdot \mathcal{O} + q \cdot \mathcal{Q}) = 2\rho \in \mathcal{A}_\rho$.

4.1. Dynamics on $\mathcal{O}_\varrho$

A geometrical condition determines the time evolution of a $su(4)$ density matrix: a solution $\rho(t)$ must be an integral curve of the mean field Hamiltonian $\overline{h}[\rho]$ or

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = [h[\rho], \rho]. \tag{27}$$

Equation (27) is a finite-dimensional Lax equation [24, 25]. It is formally the same as the time-dependent Hartree–Fock equation [39].

Dynamics may be expressed equivalently using the Poisson bracket. The Poisson bracket on $\mathcal{O}_\varrho$ is defined from the symplectic form. The bracket of two smooth real-valued functions f, g on $\mathcal{O}_\varrho$ is

$$\{f, g\}(\rho) \equiv \omega_\rho(\overline{S}_f[\rho], \overline{S}_g[\rho]). \tag{28}$$

When f is any smooth function on a coadjoint orbit, its time rate of change along a solution curve is

$$\dot{f} = \{f, \mathcal{E}\}. \tag{29}$$

For example, when $f = \lambda(S)$, the time rate of change of the observable corresponding to S along a solution curve is

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(S) = \langle\dot{\rho}, S\rangle = \omega_\rho(\overline{S}, \overline{h}[\rho]) = \{\lambda(S), \mathcal{E}\}. \tag{30}$$

The last line can be written alternatively as the derivative of $\mathcal{E}$ in the direction $\overline{S}$, $\{\lambda(S), \mathcal{E}\} = -\mathrm{d}\mathcal{E}(\overline{S})$.

4.2. Rotation group $SU(2)$

The rotation group $SU(2)$ is an embedded Lie subgroup of $SU(4)$ corresponding to the $\mathcal{D}^{(3/2)}$ irreducible representation. Each orbit of the rotation group contains a density with the quadrupole tensor in diagonal form, i.e., $q_1 = q_{-1} = 0$ and $q_2 = q_{-2}$.

Let $\mathcal{M}_\varrho$ denote the following submanifold of all densities contained in the coadjoint orbit $\mathcal{O}_\varrho$:

$$\mathcal{M}_\varrho = \{\tilde{\rho} = \rho(l, o, q) \in \mathcal{O}_\varrho \mid q_1 = q_{-1} = 0, q_{-2} = q_2\}. \tag{31}$$

Each orbit of $SU(2)$ in $\mathcal{O}_\varrho$ contains a density in the submanifold $\mathcal{M}_\varrho$. The densities $\tilde{\rho}$ in $\mathcal{M}_\varrho$ represent the density in the intrinsic rotating frame. The space of intrinsic frame densities $\mathcal{M}_\varrho$ is nine dimensional when m_1, m_2, m_3 are distinct.

When the energy function is a rotational scalar, transformation of the dynamical system to the intrinsic frame simplifies the analysis. Let $R(t) \in SU(2)$ be a smooth time-dependent rotation that transforms a solution curve of the dynamical system (27) into the submanifold of intrinsic densities. Define the time-dependent matrix

$$\Omega = -\mathrm{i}\frac{\mathrm{d}R}{\mathrm{d}t}R^{-1} = \sum_\mu \omega_\mu \mathcal{L}_\mu. \tag{32}$$

The pseudo-vector $\vec{\omega}$ corresponding to the matrix Ω is the angular velocity. Let $\tilde{\rho}(t) = R \cdot \rho \cdot R^{-1} \in \mathcal{M}_\varrho$ denote the density in the intrinsic frame. The Hamiltonian dynamical system on the coadjoint orbit, equation (27), is equivalent to the following dynamical equation on $\mathcal{M}_\varrho$:

$$\mathrm{i}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = [h_\Omega[\tilde{\rho}], \tilde{\rho}], \tag{33}$$

where $h_{\Omega}[\tilde{\rho}] = Rh[\rho]R^{-1} - \Omega$ is the $su(4)$ Routhian. When the energy function is a rotational scalar, the Hamiltonian vector field transforms covariantly under rotations, $h[\tilde{\rho}] = R \cdot h[\rho] \cdot R^{-1}$.

4.3. Range of the angular momentum

The objective of this section is to prove that the maximum value of the angular momentum on an $su(4)$ integral coadjoint orbit coincides with its maximum value in the corresponding irreducible highest weight representation. The maximum value of the squared length, $l \cdot l$, of the angular momentum is attained on any coadjoint orbit because every $su(4)$ orbit is compact and $l \cdot l$ is continuous. Such a maximum is a critical point of the smooth function $l \cdot l$, and, therefore, the corresponding Hamiltonian vector field, $X_{l \cdot l} = 2l \cdot \mathcal{L}$, vanishes at a critical point. Since $l \cdot l$ is constant on the orbits of $SU(2)$, it is sufficient to identify the critical points on the submanifold with $l \cdot l = l_0^2$ and $l_1 = l_{-1} = 0$:

$$
\begin{aligned}
0 &= [X_{l \cdot l}, \rho] = 2l_0[\mathcal{L}_0, o \cdot \mathcal{O} + q \cdot \mathcal{Q}] \\
&= 2l_0 \left(\sum_{\mu} (-1)^{\mu} o_{-\mu} (-\mu) \mathcal{O}_{\mu} + \sum_{\mu} (-1)^{\mu} q_{-\mu} (-\mu) \mathcal{Q}_{\mu} \right).
\end{aligned} \tag{34}
$$

One trivial solution is $l_0 = 0$ when the system is not rotating. Other rotating solutions require that the products $\mu o_{\mu} = 0$ and $\mu q_{\mu} = 0$ for all μ, whence the density matrix ρ is diagonal. But the diagonal matrices in the coadjoint orbit are the $4! = 24$ matrices made by permuting the entries of ϱ. The maximum angular momentum is the angular momentum of ϱ:

$$
\begin{aligned}
l_0 &= \langle \varrho, \mathcal{L}_0 \rangle = \tfrac{1}{2}(3m_1 + 4m_2 + 3m_3) \\
o_0 &= \langle \varrho, \mathcal{O}_0 \rangle = \tfrac{1}{2}(m_1 - 2m_2 + m_3) \\
q_0 &= \langle \varrho, \mathcal{Q}_0 \rangle = \tfrac{\sqrt{5}}{2}(m_1 - m_3).
\end{aligned} \tag{35}
$$

This maximal angular momentum value is the same as that found for irreducible highest weight representations of $su(4)$ with weights $[m_1, m_2, m_3]$.

4.4. Highest weight densities for $usp(4) \simeq so(5)$

A density corresponding to a $usp(4) \simeq so(5)$ highest weight vector satisfies the conditions: $l_{\mu} = o_{\mu} = 0$ for $\mu \neq 0$. The weights (n_1, n_2) of the $usp(4)$ irrep associated with such a highest weight density are the nonnegative integers $n_1 = 2(2o_0 + l_0)/5$ and $n_2 = (l_0 - 3o_0)/5$. Densities that have these special properties ($l_{\mu} = o_{\mu} = 0$ for $\mu \neq 0$) are called $usp(4)$ highest weight densities even when there is no corresponding $usp(4)$ highest weight vector. Each orbit $\mathcal{O}^{usp(4)}_{(n_1,n_2)}$ of the subgroup $USp(4)$ contains a $usp(4)$ highest weight density and it may be labelled by the real numbers n_1 and n_2. The $usp(4)$ highest weight densities contained in the $SU(4)$ coadjoint orbit $\mathcal{O}_{\varrho}$ must also satisfy the three Casimir level set equations (21).

In the $so(6)$ interacting boson model, the relevant $SU(4)$ integral coadjoint orbits correspond to symmetric irreps, $m_1 = m_3 = 0$ and $m_2 = \sigma$, a nonnegative integer. The solutions to the level set equations for the $usp(4)$ highest weight densities in the symmetric irrep case are as follows: $q_1 = q_0 = 0$, $l_0 = 2\tau$ and $o_0 = -\tau$, where $\tau = \sqrt{\sigma^2 - (2/5)|q_2|^2}$. The weights associated with these $usp(4)$ highest weight densities are $(n_1, n_2) = (0, \tau)$, where $0 \leqslant \tau \leqslant \sigma$. Note that these $usp(4)$ weights, when restricted to integers, are precisely the same as those determined by the reduction of a symmetric $su(4)$ irrep into irreps of $usp(4)$. Every

$SU(4)$ coadjoint orbit associated with a symmetric irrep is a union of these special $USp(4)$ orbits,

$$\mathcal{O}^{SU(4)}_{(0,\sigma,0)} = \bigcup_{\tau=0}^{\sigma} \mathcal{O}^{USp(4)}_{(0,\tau)}. \tag{36}$$

The algebra $so(6)$ is said to have a dynamical symmetry in the sense of the interacting boson model when the energy is a function of the quadratic Casimir functions of the subalgebra chain $so(6) > so(5) > su(2)$,

$$\mathcal{E}_{\mathrm{IBM}}(\rho) = A\mathcal{C}^{(2)}_{usp(4)} + Bl \cdot l, \tag{37}$$

where A, B are real constants and the quadratic $usp(4) \simeq so(5)$ Casimir function is $\mathcal{C}^{(2)}_{usp(4)} = l \cdot l + o \cdot o$. The Routhian for this energy is

$$h_\Omega[\tilde{\rho}] = 2A(l \cdot \mathcal{L} + o \cdot \mathcal{O}) + 2Bl \cdot \mathcal{L} - \Omega. \tag{38}$$

Every $usp(4)$ highest weight density is in equilibrium for this Routhian, i.e., $[h_\Omega[\tilde{\rho}], \tilde{\rho}] = 0$, for any $usp(4)$ highest weight density $\tilde{\rho} = l_0\mathcal{L}_0 + o_0\mathcal{O}_0 + q_0\mathcal{Q}_0 + q_2(\mathcal{Q}_{-2} + \mathcal{Q}_{-2})$. This commutator vanishes for $usp(4)$ highest weight densities when the components of the angular velocity are $\omega_1 = \omega_{-1} = 0$ and $\omega_0 = 2(A + B)l_0 - Ao_0$. If the $usp(4)$ highest weight density lies on the symmetric irrep orbit $\mathcal{O}^{SU(4)}_{(0,\sigma,0)}$, then, for $0 \leqslant l_0 \leqslant 2\sigma$,

$$q_0 = 0, \qquad o_0 = -l_0/2, \qquad q_2 = \pm\sqrt{(5/2)\left(\sigma^2 - l_0^2/4\right)}. \tag{39}$$

4.5. Relationship between $SU(4)$ and $USp(4)$ mean field theories

When the energy function $\mathcal{E}(l, o)$ is independent of the quadrupole moment q, the Hamiltonian dynamical system on an $SU(4)$ coadjoint orbit determines a Hamiltonian dynamical system on a $USp(4)$ coadjoint orbit. Let $\pi : su(4) \to usp(4)$ denote the projection of $su(4)$ onto its Lie subalgebra $usp(4)$, $\pi(\rho) = \rho^{usp(4)}$, where $\rho = l \cdot \mathcal{L} + o \cdot \mathcal{O} + q \cdot \mathcal{Q} \in su(4)$ projects onto $\rho^{usp(4)} = l \cdot \mathcal{L} + o \cdot \mathcal{O} \in usp(4)$.

For an energy function $\mathcal{E}(l, o)$, the corresponding Hamiltonian vector field is an element of the subalgebra $usp(4)$ and depends only on the projection of the density, $h[\rho] = h[\rho^{usp(4)}] \in usp(4)$. The $SU(4)$ dynamical system is

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho^{usp(4)} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}q \cdot \mathcal{Q} = [h[\rho^{usp(4)}], \rho^{usp(4)}] + [h[\rho^{usp(4)}], q \cdot \mathcal{Q}]. \tag{40}$$

The first terms on the left and right sides are in the $usp(4)$ subalgebra and second are in the complementary subspace spanned by the $\mathcal{Q}_\mu$. After projection the system simplifies to a Hamiltonian Lax dynamical system on $USp(4)$ orbits,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho^{usp(4)} = [h[\rho^{usp(4)}], \rho^{usp(4)}]. \tag{41}$$

An integral curve of the $su(4)$ dynamical system factors through the projection to an integral curve of the $usp(4)$ dynamical system. In a recent paper, $usp(4)$ mean field theory was developed and analytic equilibrium solutions were reported for certain energy functions of interest [11]. To finish solving the $su(4)$ dynamical system, the linear differential equation

$$\mathrm{i}\frac{\mathrm{d}q}{\mathrm{d}t} \cdot \mathcal{Q} = [h[\rho^{usp(4)}(t)], q \cdot \mathcal{Q}] \tag{42}$$

must be solved for $q(t)$.

A trivial solution to this differential equation is $q = 0$. Therefore, any solution to the $usp(4)$ dynamical system, equation (41), lifts to a solution to the $su(4)$ dynamical system. Such

Table 3. Critical points of the $usp(4)$ quadratic Casimir in $\mathcal{O}_\varrho$.

n_1	n_2
$m_1 + m_3$	m_2
$m_1 - m_3$	$m_2 + m_3$
$m_1 - m_3$	m_3
$m_3 - m_1$	$m_1 + m_2$
$m_3 - m_1$	m_1

$su(4)$ solutions lie in the subspace $usp(4)$ of all $su(4)$ densities with $q = 0$. This subspace is invariant with respect to the $USp(4)$ action. On the subspace $usp(4)$, the three $su(4)$ Casimirs are functionally dependent:

$$8\mathcal{C}^{(6)} - \mathcal{C}^{(2)}(6\mathcal{C}^{(4)} - (\mathcal{C}^{(2)})^2) = 0. \tag{43}$$

The point ϱ is not typically in $usp(4)$, but it is when $m_1 = m_3$.

4.6. Range of the $usp(4) \simeq so(5)$ quadratic Casimir

If the maximum value of the $usp(4)$ Casimir on an $SU(4)$ coadjoint orbit is attained for the density ρ, then the vector field $l \cdot \mathcal{L} + o \cdot \mathcal{O}$ is zero at ρ. Since this Casimir function is constant on orbits of the subgroup $USp(4)$, it is sufficient to locate the critical points ρ on $USp(4)$ orbit representatives which satisfy $l_\mu = 0$ and $o_\mu = 0$ for $\mu \neq 0$:

$$\begin{aligned} 0 &= [l_0\mathcal{L}_0 + o_0\mathcal{O}_0, q \cdot \mathcal{Q}] \\ &= (2l_0 - o_0)(q_2\mathcal{Q}_{-2} - q_{-2}\mathcal{Q}_2) + (l_0 + 2o_0)(q_{-1}\mathcal{Q}_1 - q_1\mathcal{Q}_{-1}). \end{aligned} \tag{44}$$

Neither $2l_0 = o_0$ nor $l_0 = -2o_0$ are consistent with ρ satisfying the three $su(4)$ Casimir identities. Therefore the density ρ must be diagonal. The maximum value of the $usp(4)$ Casimir is attained at the density ϱ. The weights (n_1, n_2) of the $usp(4)$ representations that correspond to the critical points of the $usp(4)$ quadratic Casimir are given by

$$n_1 = \langle \rho, E_{11} - E_{22} + E_{33} - E_{44} \rangle \qquad n_2 = \langle \rho, E_{22} - E_{33} \rangle. \tag{45}$$

At ϱ the $usp(4)$ weights are $n_1 = m_1 + m_3$ and $n_2 = m_2$. The $usp(4)$ weights of critical points of the $usp(4)$ Casimir are listed in table 3.

4.7. Energy function

The energy function $\mathcal{E}(\rho)$ is a real-valued function defined on the dual space $su(4)^*$. This function is assumed to be invariant under rotations, $\mathcal{E}(R \cdot \rho \cdot R^{-1}) = \mathcal{E}(\rho)$ for all $R \in SU(2)$. As a consequence, the angular momentum vector is constant along each solution curve.

For example, a simple real-valued smooth rotationally invariant energy function is

$$\mathcal{E}(\rho) = A\{(l \cdot l) + B(l \cdot [q \times l]^{(1)}) + C(l \cdot [q \times o]^{(1)})\}, \tag{46}$$

where A, B, C are real constants. Table 2 gives the Hamiltonian vector field associated with each term in $\mathcal{E}(\rho)$. A density $\tilde{\rho} \in \mathcal{M}_\varrho$ is a rotating equilibrium solution when it commutes with $h_\Omega[\tilde{\rho}]$. The requirement that the commutator matrix $[\tilde{\rho}, h_\Omega[\tilde{\rho}]]$ lies in $\mathcal{M}_\varrho$ determines the angular velocity Ω. The density $\tilde{\rho}$ also satisfies the three Casimir equations (21).

For the symmetric irrep orbits, $m_1 = m_3 = 0$ and $m_2 = \sigma$, rotating equilibrium solutions for this energy function include a class that are analytically solvable and correspond to rotation about a single principal axis, $\omega_{\pm 1} = l_{\pm 1} = 0$. The density components l_0, o_μ, q_μ of the solution are real, and

$$\begin{aligned} &o_0 = -l_0/2 \qquad o_1 = 0 \qquad |o_2|^2 = 5l_0(2\sigma - l_0)/8 \\ &o_3 = 0 \qquad q_0 = \sqrt{5}(2\sigma - l_0)/2 \qquad q_2 = -o_2. \end{aligned} \tag{47}$$

The solutions are limited to a maximum value of l_0 since $|o_2|^2$ is nonnegative, $0 \leqslant l_0 \leqslant 2\sigma$. The energies of these rotating equilibrium densities are

$$\mathcal{E}(\tilde{\rho})/A = l_0^2 + \chi l_0^2(2\sigma - l_0), \tag{48}$$

for $\chi = -B/\sqrt{2} - 2C/\sqrt{7}$. Equivalently the energy may be expressed in the form

$$\mathcal{E}(\tilde{\rho})/A = \frac{l_0^2}{2\mathcal{I}}, \tag{49}$$

where the moment of inertia is

$$\mathcal{I} = \frac{1}{2[1 + \chi(2\sigma - l_0)]}. \tag{50}$$

When χ is positive, the moment of inertia is a monotonically increasing function of the angular momentum.

5. Discussion

In the case when weak dynamical symmetry is present but dynamical symmetry is strongly broken, algebraic mean field theory appears to be the only practical theoretical tool to provide a physical understanding of a complex quantum system. Even if a brute force computer calculation of the system's spectrum is feasible, the results are often unilluminating. Mean field theory exploits the underlying weak dynamical symmetry to reveal the system's fundamental simplicity. However, only the physical observables corresponding to the algebra elements show this simplicity. Algebraic mean field theory makes no predictions for the observables corresponding to the degrees of freedom that are responsible for the breaking of dynamical symmetry. Such degrees of freedom are not included in the Lie algebra.

Mean field theory for integral orbits is closely related to the Kirillov program to derive irrep properties directly from orbit data [32]. This paper derives an analytic formula, equation (48), in the mean field approximation for a particular energy function, equation (46). The paper also reports formulae, equation (47), for the expectations of other $su(4)$ algebra elements. Such formulae provide an immediate understanding of the dependence of physical observables on the angular momentum and the $su(4)$ highest weight data. The eigenvalue spectrum in irreducible representations for the Hamiltonian operator associated with this energy function requires numerical diagonalization.

In a future paper I plan to derive mean field theory for the $u(6)$ algebra and apply it to understand quantum phase transitions associated with $u(6)$ subalgebras, namely, $so(6)$–$u(5)$ and $u(5)$–$su(3)$ [16–18]. The mean field results of this paper for $so(6) \simeq su(4)$ and prior papers about $so(5)$ [11] and $su(3)$ [2, 4] are relevant to an investigation of quantum phase transitions in the $u(6)$ interacting boson model.

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